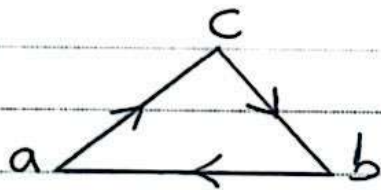
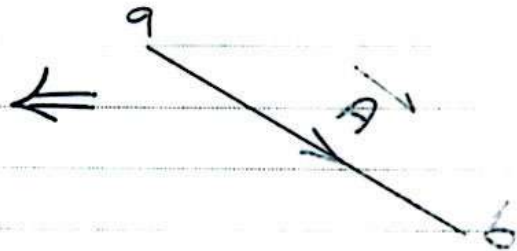


Lecture: 1

Chapter: 1

vectors analysis:

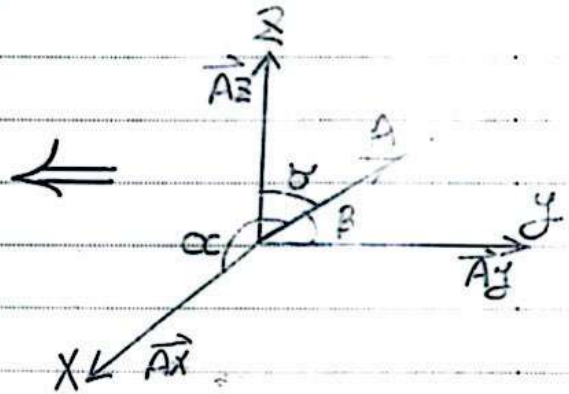
$$\vec{ab} = |\vec{ab}| = |\vec{A}|$$



$$\Rightarrow \vec{ac} + \vec{cb} = \vec{ba}$$

$$\vec{A} = (A_x, A_y, A_z)$$

$$= A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$



$$* \cos \alpha = \frac{A_x}{\|\vec{A}\|}$$

$$* \cos \beta = \frac{A_y}{\|\vec{A}\|}$$

$$* \cos \gamma = \frac{A_z}{\|\vec{A}\|}$$

$$\Rightarrow \hat{A} = \frac{\vec{A}}{\|\vec{A}\|} = \frac{A_x \vec{i} + A_y \vec{j} + A_z \vec{k}}{\|\vec{A}\|} =$$

"مقبول الوحدة"

$$= \frac{A_x \vec{i}}{\|\vec{A}\|} + \frac{A_y \vec{j}}{\|\vec{A}\|} + \frac{A_z \vec{k}}{\|\vec{A}\|}$$

$$= \cos \alpha \vec{i} + \cos \beta \vec{j} + \cos \gamma \vec{k} \quad \#$$

$$*|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\frac{A^2}{A^2} = \frac{A_x^2}{A^2} + \frac{A_y^2}{A^2} + \frac{A_z^2}{A^2}$$

$$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$$

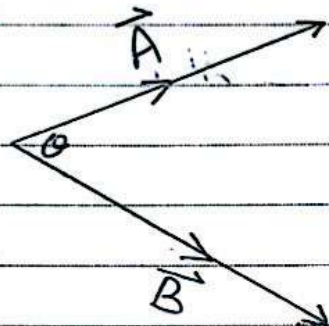
$$\vec{P_1 - P_2} = \vec{P_2 - P_1}$$

$$= (-3, 1, 1)$$

$$= -3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$P_2 (-2, 3, 4)$

$P_1 (1, 2, 3)$



الضرب التناهي الضيق

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

$$* \mathbf{i} \cdot \mathbf{j} = 1 \times 1 \cos 90 = 0$$

$$* \mathbf{j} \cdot \mathbf{j} = 1 \times 1 \cos 0 = 1$$

$$\Rightarrow \vec{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$$

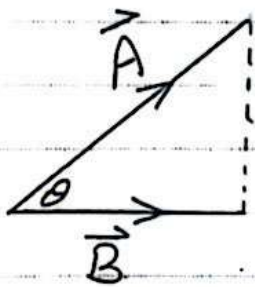
$$\Rightarrow \vec{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

$$* \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \cdot \vec{B} = 2 \text{ zero?}$$

$$* \vec{A} = 0 \text{ or } \vec{B} = 0$$

$$* \vec{A} \perp \vec{B}$$

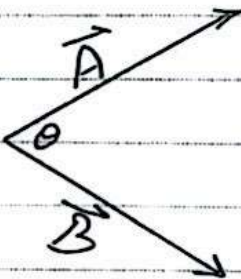


$$A \cdot B = A \frac{\vec{A} \cdot \vec{B}}{AB}$$

← مسقط $\vec{A}$ على $\vec{B}$

$$\cos \theta = \frac{\text{المسقط}}{\text{الوتر}}$$

$$A \cos \theta = \text{المسقط}$$



* الضرب الاتجاهي الثنائي

$$A \wedge B = AB \sin \theta$$

$$* i \wedge j = k$$

$$* i \wedge i = 0$$

$$\Rightarrow \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

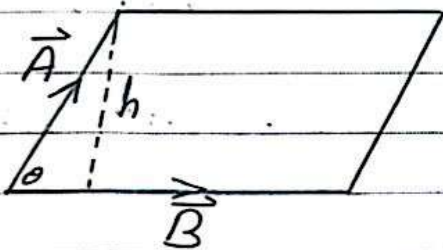
$$\Rightarrow \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$* \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$* \vec{A} = 0 \text{ or } \vec{B} = 0$$

$$\vec{A} \times \vec{B} = \text{Zero?} \quad * \vec{A} \parallel \vec{B}$$

$$* \text{صيف أو عمود} = \text{صيف}$$



مساحة المتوازي = طول القاعدة $\times$ الارتفاع

$$\text{Area} = B h \quad \leftarrow \text{القاعدة} \times \text{الارتفاع}$$

$$\sin \theta = \frac{h}{A}$$

$$h = A \sin \theta$$

$$\text{Area} = B A \sin \theta$$

$$= |\vec{A} \times \vec{B}| \quad \#$$

* "الضرب التبادلي القياسي"

$$A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$* A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) = \text{Volume}$$

$$\Rightarrow \vec{A} \cdot \vec{A} = |\vec{A}|^2$$

$$\Rightarrow \vec{A} \times \vec{A} = 0$$

* $A=0$ or $B=0$ or $C=0$

$A \cdot (B \times C) = \text{Zero?}$ * على الأقل متجهيه متوازيين

* اذا وقع الثلاث متجهات في نفس المستوى

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \text{حجم متوازي السطوح}$$

* الرضرب التبادلي الاتجاهي *

$$A_n(B_n C) = (A \cdot C)B - (B \cdot C)A$$

أثبت أنه: $* A_n(B_n C) + B_n(C_n A) + C_n(A_n B) = 0$

$$[(A \cdot C)B - (B \cdot C)A] + [(B \cdot A)C - (C \cdot A)B] + [(C \cdot B)A - (A \cdot B)C] = \underline{\underline{Zero}}$$

متجهات الوحدة

$$i_n j = k$$

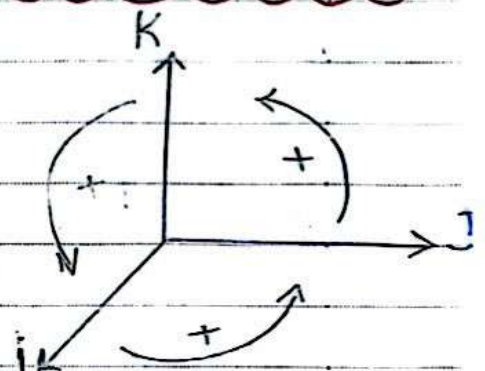
$$i_n i = 0$$

$$j_n k = i$$

$$j_n j = 0$$

$$k_n i = j$$

$$k_n k = 0$$



Lecture: 2

$$* A \cap (B \cap C) = (A \cap C) \cap B = (A \cap B) \cap C$$

EX: $a \cap x = a \cap b$, $a \cdot x = b$, $a \cdot b = \text{Zero}$

← x المطلوب

$$a \cap (a \cap x) = a \cap (a \cap b) \quad \text{بالضرب في } (a)$$

$$(a \cdot x) \cdot a - (a \cdot a) \cdot x = (a \cdot b) \cdot a - (a \cdot a) \cdot b$$

$\swarrow$ $\searrow$ $\swarrow$ $\searrow$
 b Zero Zero Zero

$$ba - a^2 x = -a^2 b$$

$$ba + a^2 b = a^2 x$$

$$\therefore x = \frac{ba + a^2 b}{a^2} = \frac{b + ab}{a} = \frac{ba}{a^2} + b \quad \text{X}$$

Ex: $x \cdot a = 0$, $b - x = \lambda a$ $\Rightarrow \lambda = \frac{a \cdot b}{b^2}$

$\Rightarrow x = \frac{(a \cdot b)a}{a^2} \leftarrow \text{أثبت أن}$

* $b - x = \lambda a$ بالضرب $x(a)$

$a \cdot b - \cancel{a x} = \lambda a \cdot a$
zero

$a \cdot b = \lambda a^2 \quad \therefore \lambda = \frac{a \cdot b}{a^2} \quad \# \quad \boxed{1}$

* $b - x = \lambda a$

$b - x = \frac{a \cdot b}{a^2} a$

$\lambda = b - \frac{a \cdot b}{a^2} a$ "توحيد المقام"

$x = \frac{a^2 b}{a^2} - \frac{a \cdot b}{a^2} a = \frac{a^2 b - (a \cdot b)a}{a^2}$

$x = \frac{(a \cdot a)b - (a \cdot b)a}{a^2}$

$x = \frac{(a \cdot b)a}{a^2}$

2

← إيجاد الزاوية بين المتجهين ..

EX: $A = (2, 2, -1)$; $B = (3, 12, 4)$

اثبت أن $\theta = \cos^{-1}\left(\frac{3}{2}\right)$

$$A \cdot B = AB \cos \theta$$

$$\cos \theta = \frac{A \cdot B}{AB} = \frac{6 + 24 - 4}{\sqrt{4+4+1} \times \sqrt{9+144+16}} = \frac{26}{39}$$

$$= \frac{2}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

~~لا يوجد اثبات~~

EX: $A = (-2, 2, 1)$, $B = (3, 1, 0)$ ← مسقط A على B ..

اثبت أن مسقط A على B $= \frac{+4}{\sqrt{10}}$

$$\underline{\hat{B}} = \frac{B}{|B|} = \frac{(3, 1, 0)}{\sqrt{9+1}} = \frac{(3, 1, 0)}{\sqrt{10}}$$

$$\text{المسقط} = \underline{A} \cdot \underline{\hat{B}} = (-2, 2, 1) \cdot \frac{(3, 1, 0)}{\sqrt{10}} = \frac{-6+2}{\sqrt{10}}$$

$$= \frac{-4}{\sqrt{10}}$$

~~لا يوجد اثبات~~

Ex: $A(1, 0, 1)$, $B(2, 3, 4)$, $C(-1, 1, -2)$

$\therefore \frac{27}{7} = BC$ فان سقط BA على BC

$$BA = A - B = (1, 0, 1) - (2, 3, 4) = (-1, -3, -3)$$

$$BC = C - B = (-1, 1, -2) - (2, 3, 4) = (-3, -2, -6)$$

$$\hat{BC} = \frac{BC}{|BC|} = \frac{(-3, -2, -6)}{\sqrt{9+36+4}} = \frac{(-3, -2, -6)}{7}$$

$$\text{المسقط} = (-1, -3, -3) \cdot \frac{(-3, -2, -6)}{7}$$

$$= \frac{3+6+18}{7} = \frac{27}{7} \quad \#$$

ناتج ثابت صحيح .

Ex: $|A| = 7.4$ بن زاوية 250° عكس اتجاه عقارب الساعة

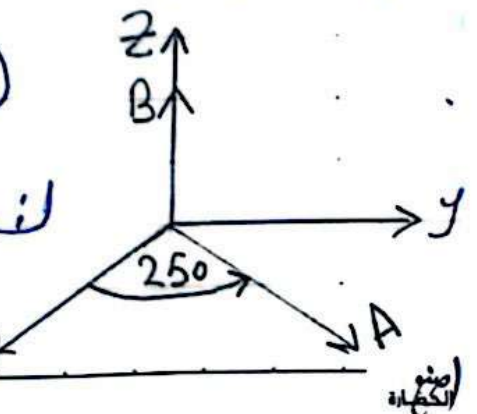
$|B| = 6 \Rightarrow$ اتجاه المحور z $B = (0, 0, 6)$

$$A = 7.4 \cdot \cos(250^\circ) + 7.4 \cdot \sin(250^\circ)$$

$A \cdot B = 0 \rightarrow$ لانه الزاوية المحصورة هي 90°

$$|A \wedge B| = AB \sin \theta = 7.4 \times 6 \times \sin 90^\circ$$

$$= 37$$



$$A = (-2,53, -6,95, 0)$$

$$B = (0, 0, 5)$$

$$A \wedge B = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ -2,53 & -6,95 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= -34,75j + 12,65i$$

$$\tan \theta = \frac{j}{i} = \frac{-12,65}{34,75}$$

$$\therefore \theta = \underline{\underline{20^\circ}} \quad \#$$

Chapter 2

"مجموع القوى في الفراغ"

$\underline{F} = 13$

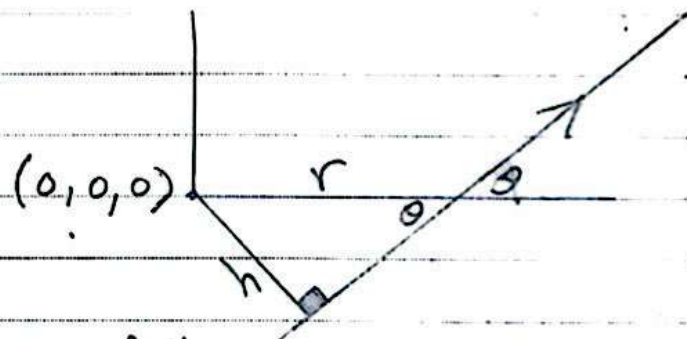
$A_1 (4, -2, 6)$ $A_2 (8, 1, -6)$

$\underline{F} = \frac{\underline{F}}{F}$ $F = 13$

$$\underline{F} = F \cdot \underline{\hat{F}}$$

$$\Rightarrow A_1 A_2 = A_2 - A_1 = (4, 3, -12)$$

$$\Rightarrow \underline{F} = F \cdot \underline{\hat{F}} = 13 \frac{(4, 3, -12)}{\sqrt{16+9+144}} = (4, 3, -12)$$



$$\sin \theta = \frac{h}{r}$$

$$\therefore h = r \sin \theta$$

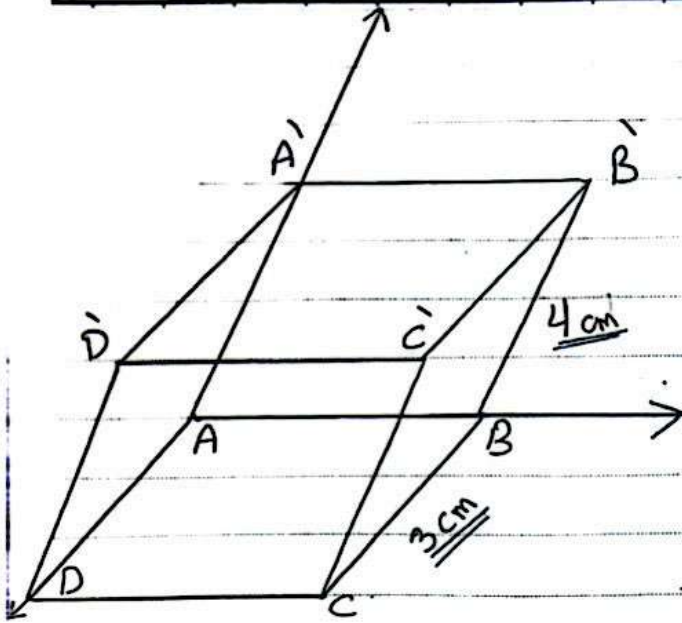
الزخم $G = F h$

$$= F r \sin \theta$$

$$\therefore G = |r \wedge F|$$

EX:3, P:49

lecture:3



$F_1 \rightarrow 5 \cdot \underline{CB'}$, $F_2 \rightarrow 10 \cdot \underline{AC}$

* $A(0,0,0)$ * $B(0,0,3)$

* $A'(0,0,4)$ * $B'(0,0,4)$

* $C(3,4,0)$ * $C'(3,4,4)$

* $D(3,0,0)$ * $D'(3,0,4)$

* $(G_{F_1})_A \Rightarrow$ نقطة F_1 حول الزاوية

$$\underline{G_1} = r \wedge F_1$$

$$* \underline{CB'} = B' - C$$

$$= AC \wedge F_1$$

$$= (0,4,4) - (3,4,0)$$

$$= (-3,0,4)$$

$$* \underline{AC} = C - A$$

$$= (3,4,0)$$

$$* \underline{\hat{CB'}} = \frac{(-3,0,4)}{5}$$

$$* (G_1)_A = \begin{vmatrix} i & j & k \\ 3 & 4 & 0 \\ -3 & 0 & 4 \end{vmatrix}$$

$$* \underline{\hat{F_1}} = 5 \times \frac{(-3,0,4)}{5}$$

$$\underline{\hat{F_1}} = (-3,0,4)$$

$$= (16, -12, 12) \#$$

$$* (G_{F_2})_D \Rightarrow D \text{ حول النقطة } D$$

$$\begin{aligned} * (G_2)_D &= r \cap F_2 \\ &= DA' \cap F_2 \end{aligned}$$

$$\begin{aligned} * DA' &= A' - D \\ &= (0, 0, 4) - (3, 0, 0) \\ &= (-3, 0, 4) \end{aligned}$$

$$* (G_2)_D = \begin{vmatrix} i & j & k \\ -3 & 0 & 4 \\ 6 & 8 & 0 \end{vmatrix}$$

$$= (-32, 24, -24) \neq$$

$$* (G_{F_2})_A \Rightarrow A \text{ حول النقطة } A$$

$$\begin{aligned} * AA' &= A' - A \\ &= (0, 0, 4) \end{aligned}$$

$$\begin{aligned} * \underline{AC} &= C - A \\ &= (3, 4, 4) - (0, 0, 4) \\ &= (3, 4, 0) \end{aligned}$$

$$* \underline{AC} = \frac{(3, 4, 0)}{5}$$

$$\begin{aligned} * \underline{F_2} &= 10 \times \frac{(3, 4, 0)}{5} \\ &= (6, 8, 0) \end{aligned}$$

$$\begin{aligned} (G_2)_A &= r \cap F_2 \\ &= AA' \cap F_2 \end{aligned}$$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & 4 \\ 6 & 8 & 0 \end{vmatrix}$$

$$= (-32, 24, 0) \neq$$

نقطة
* العزم المصحل حول المحور

$$\begin{aligned}\underline{G} &= (G_1)_A + (G_2)_A \\ &= (16, -12, 12) + (-32, 24, 0) \\ &= (-16, 12, 12) \quad = |\underline{G}| = \sqrt{(16)^2 + (12)^2 + (12)^2} \\ &= 4(-4, 3, 3) \quad = \underline{\underline{4\sqrt{34}}} \\ &\quad \# \end{aligned}$$

* العزم المصحل حول محور AC

* العزم حول محور = العزم حول نقطة على نفس المحور

$$* \underline{AC} = \underline{C} - \underline{A}$$

$$= (3, 4, 4)$$

$$\underline{\hat{AC}} = \frac{(3, 4, 4)}{\sqrt{9+16+16}}$$

$$= \frac{(3, 4, 4)}{\sqrt{41}}$$

$$G_e = \underline{\hat{AC}} \cdot \underline{G}$$

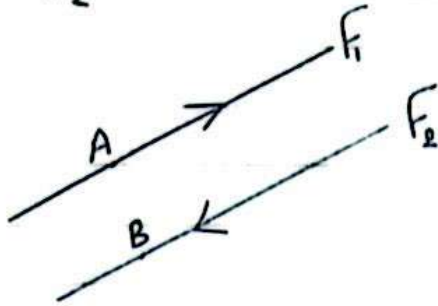
$$= \frac{4}{\sqrt{41}} (-12 + 12 + 12)$$

$$= \frac{48}{\sqrt{41}} \quad \#$$

lecture .4

"عزيم الزواج"

$$F_1 = F_2$$



$$\begin{aligned} \underline{G} &= \underline{r} \wedge \underline{F} \\ &= \underline{BA} \wedge \underline{F} \end{aligned}$$

EX:4 , P:55

$$\underline{F} = 120 \cdot \underline{AC}$$

$$* \underline{\hat{AC}} = \underline{C} - \underline{A}$$

$$= (0, 0, 4) - (8, 8, 0)$$

$$= (-8, -8, 4) \\ \underline{\underline{\frac{(-8, -8, 4)}{\sqrt{64+64+16}}}}} = \underline{\underline{\frac{(-8, -8, 4)}{12}}}$$

$$\underline{F} = 120 \underline{\hat{AC}}$$

$$= 120 \cdot \frac{(-8, -8, 4)}{12} = (-80, -80, 40) \neq$$

$$* \underline{M_P} = \underline{PA} \wedge \underline{F}$$

$$* \underline{PA} = \underline{A} - \underline{P}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 14 & -10 \\ -80 & -80 & 40 \end{vmatrix}$$

$$\begin{aligned} &= (8, 8, 0) - (8, -6, 10) \\ &= \underline{\underline{(2, 14, -10)}} \end{aligned}$$

$$= (-240, 720, 960) \neq$$

* اللولبية :

$$P = \frac{F \cdot M}{F^2}$$

↓
"درجة اللولبية"

$$\Rightarrow F \cdot M \neq 0 \Rightarrow \text{"تؤثر إلى لولبية"}$$

$$\Rightarrow F \cdot M = 0 \Rightarrow \text{"تؤثر إلى قوة وحيدة"}$$

← معادلة محور اللولبية :

$$* M' = M - r \wedge F, \quad M' = PF$$

$$* r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$* \therefore PF = M - r \wedge F$$

$$M = PF + r \wedge F \quad \#$$

← معادلة محور القوة الوحيدة :

$$* \underline{M} = r \wedge F$$

$$M' = P\vec{F}$$

* نرمز اللولبية :

$$P = \frac{F \cdot M}{F^2}$$

* درجة اللولبية :

EX: 8

$$\vec{F} = -20\mathbf{i} + 10\mathbf{j} + 15\mathbf{k}$$

$$A(0,0,0), B(3,4,0), C(3,4,4)$$

$$\vec{BC} = \vec{r}_{CB} = \vec{r}_C - \vec{r}_B = (0,0,4) = 4\hat{k}$$

$$\underline{\vec{AB}} = \frac{(3,4,0)}{\sqrt{9+16}} = \frac{1}{5}(3,4,0)$$

$$\Rightarrow G_{AB} = \begin{vmatrix} 0 & 0 & 4 \\ 3 & 4 & 0 \\ -20 & 10 & 15 \end{vmatrix} = 440$$

Ex:

نقاط قوت متاربع f ...
 $C(0,0,3) \xrightarrow{\text{قوت من}} 0x$

$A(1,0,0) \xrightarrow{\text{قوت من}} 0y$, $B(0,2,0) \xrightarrow{\text{قوت من}} 0z$

اثبت أن المجموعة تكافؤ لولبية، أوجد درجتها، ومعادلة محورها.

$$* \vec{F}_1 = f j \quad * \vec{F}_2 = f k \quad * \vec{F}_3 = f i$$

$$\text{مطلوب} \vec{F} = f(i, j, k) \#$$

$$* \vec{M}_1 = \vec{r}_{AO} \wedge \vec{F} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & f & 0 \end{vmatrix} = f k$$

$$* \vec{M}_2 = \vec{r}_{BO} \wedge \vec{F} = \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 0 & 0 & f \end{vmatrix} = 2f i$$

$$* \vec{M}_3 = \vec{r}_{CO} \wedge \vec{F} = \begin{vmatrix} i & j & k \\ 0 & 0 & 3 \\ f & 0 & 0 \end{vmatrix} = 3f j$$

$$\text{المطلوب} \vec{M} = 2f i + 3f j + f k : \#$$

$$\vec{F} \cdot \vec{M} = 2f^2 \hat{i} + 3f^2 \hat{j} + f^2 \hat{k} \\ = 6f^2 \#$$

$\Rightarrow 6f^2 \neq 0 \therefore$ تَوَرُّدٌ إِلَى لَوَّلِيَّةٍ

$$P = \frac{\vec{F} \cdot \vec{M}}{F^2} = \frac{6f^2}{3f^2} = 2 \# \leftarrow \text{درجۃ اللولبيّة}$$

$$\vec{M}' = P \cdot \vec{F} = 2f (i + j + k) \# \leftarrow \text{عزم اللولبيّة}$$

معادلة المحاور

$$\vec{M} = \vec{M}' + \vec{r} \wedge \vec{F}$$

$$\vec{r} \wedge \vec{F} = \vec{M} - \vec{M}'$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ f & f & f \end{vmatrix} = f(j-k)$$

$$f(y-z)\vec{i} - f(x-z)\vec{j} + f(x-y)\vec{k} = f\vec{j} - f\vec{k}$$

$$\begin{aligned} * y-z &= 0 & * x-z &= 1 & * x-y &= -1 \end{aligned}$$

$$\Rightarrow \therefore x+1 = y = z$$

$$\Rightarrow \frac{x+1}{f} = \frac{y}{f} = \frac{z}{f} \neq$$

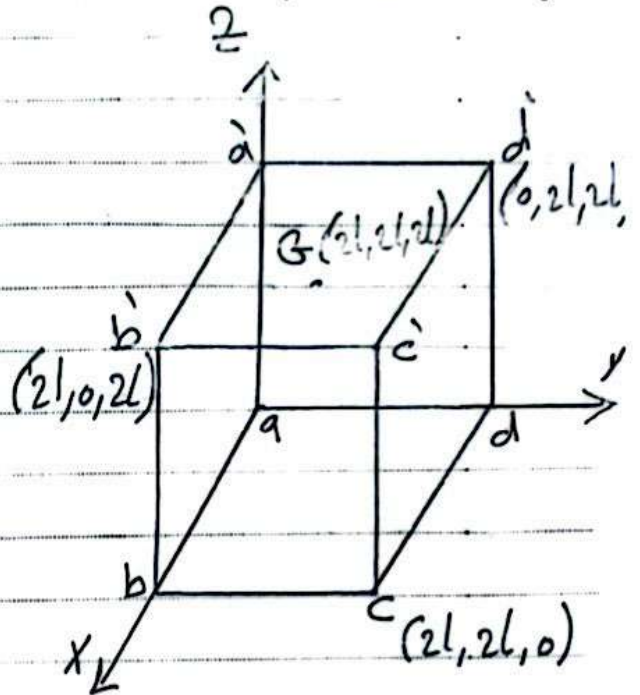
ALADID

مكتبة

قوتان F ، مكعب طول ضلعه $2l$. أوجد عزم الالتواء عند المركز

$$F_1 \rightarrow f, \underline{cd}$$

$$F_2 \rightarrow f, \underline{ab}$$



$$\underline{F}_1 = f \cdot \frac{(2l, 0, 2l)}{2\sqrt{2}l} = \frac{f}{\sqrt{2}} (\underline{i} + \underline{k})$$

$$\underline{F}_2 = f \cdot \frac{(2l, 0, 2l)}{2\sqrt{2}l} = \frac{f}{\sqrt{2}} (\underline{i} + \underline{k})$$

$$\underline{F} = \frac{2f}{\sqrt{2}} \underline{k} = \sqrt{2} f \underline{k}$$

$$\underline{M}_1 = \underline{r} \wedge \underline{F}_1 = \underline{Gc} \wedge \underline{F} = \frac{f}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & -2l \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \frac{-2lf}{\sqrt{2}} \underline{j}$$

$$\underline{M}_2 = \underline{r} \wedge \underline{F}_2 = \underline{Ga} \wedge \underline{F} = \frac{f}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2l & -2l & 2l \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \frac{f}{\sqrt{2}} (-2l \underline{i} - 0 \underline{j} + 2l \underline{k}) = \frac{-2lf}{\sqrt{2}} \underline{i} + \frac{2lf}{\sqrt{2}} \underline{k}$$

Ex:

قوة $F \leftarrow 6N \leftarrow$ تصنع زاوية متساوية

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$3 \cos^2 \alpha = 1 \Rightarrow \cos \alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \underline{\hat{A}} = \frac{1}{\sqrt{3}} \underline{i} + \frac{1}{\sqrt{3}} \underline{j} + \frac{1}{\sqrt{3}} \underline{k}$$

$$\Rightarrow \underline{F} = 6 \cdot \underline{\hat{A}} = \frac{6}{\sqrt{3}} (\underline{i} + \underline{j} + \underline{k}) = \underline{\underline{2\sqrt{3}(\underline{i} + \underline{j} + \underline{k})}}$$

$\leftarrow$ أوجد العزم حول نقطة O .

إذا كانت نقطة تأثير القوى $C(3, 5, -2)$

$$\underline{M} = \underline{r} \wedge \underline{F} \Rightarrow \underline{r} = \underline{OC} = (3, 5, -2)$$

$$= 2\sqrt{3} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 5 & -2 \\ 1 & 1 & 1 \end{vmatrix} = (14\sqrt{3} \underline{i} - 10\sqrt{3} \underline{j} - 4\sqrt{3} \underline{k})$$

$$|\underline{M}| = \sqrt{(14\sqrt{3})^2 + (10\sqrt{3})^2 + (4\sqrt{3})^2} = 6\sqrt{26}$$

$$= \underline{\underline{2\sqrt{334}}}$$

Ex:

1 1

قوتان $\underline{F} = 2\hat{i} - 3\hat{j} \rightarrow A(-6, 0, 2)$ ←

$-\underline{F} = -2\hat{i} + 3\hat{j} \rightarrow B(1, -2, 0)$

$M_1 = \underline{r} \wedge \underline{F}_1 \Rightarrow \underline{r} = \underline{BA} = (-6, 2, 2)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 2 & 2 \\ 2 & -3 & 0 \end{vmatrix} = 6\hat{i} + 4\hat{j} + 14\hat{k}$

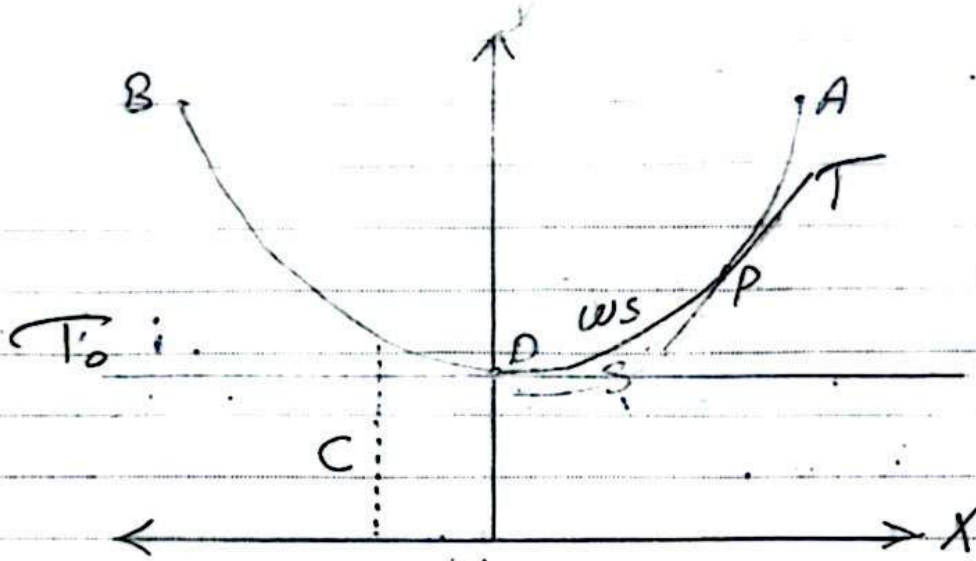
$|\underline{M}| = \sqrt{36 + 16 + 196} = \underline{\underline{2\sqrt{62}}}$

* جيوب تمام الزاياه :

1) $\cos \alpha = \frac{6}{2\sqrt{62}} = \frac{3}{\underline{\underline{\sqrt{62}}}}$

2) $\cos \beta = \frac{4}{2\sqrt{62}} = \frac{2}{\underline{\underline{\sqrt{62}}}}$

3) $\cos \gamma = \frac{14}{2\sqrt{62}} = \frac{7}{\underline{\underline{\sqrt{62}}}}$



$S \Rightarrow$ طول القوس $PD \Rightarrow$ طول الجذر المعلق من السلسلة

$\omega \Rightarrow$ مسافة التمدد

$T \Rightarrow$ الشد في الجذر المعلق من السلسلة

$\mu \Rightarrow$ زاوية ميل العمود على الأفقي

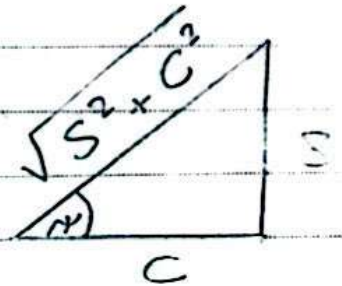
$$T \cos \mu = T_0 = \omega c \quad \text{المركبة الأفقية للشد}$$

$$T \sin \mu = \omega s \quad \text{المركبة الرأسية للشد}$$

المعادلة الذاتية :-

$$\frac{\cancel{r} \sin \mu}{\cancel{r} \cos \mu} = \frac{\cancel{w} s}{\cancel{w} c}$$

$$\tan \mu = \frac{s}{c}$$



$$\therefore s = c \tan \mu$$

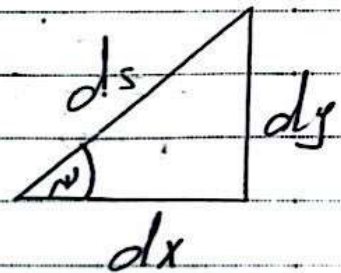
$$\sin \mu = \frac{s}{\sqrt{s^2 + c^2}}$$

$$\frac{ds}{d\mu} = c \sec^2 \mu$$

$$\cos \mu = \frac{c}{\sqrt{s^2 + c^2}}$$

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\frac{ds}{dx} = \sqrt{1 + y'^2}$$



$$\sin \mu = \frac{dy}{ds}$$

$$\cos \mu = \frac{dx}{ds}$$



المعادلة التفاضلية

$$\frac{dy}{dx} = \tan \psi, \quad S = c \tan \psi$$

$$\therefore y' = \frac{S}{c}$$

$$\therefore \frac{dy'}{dx} = \frac{1}{c} \frac{ds}{dx}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + y'^2}$$

$$\therefore \frac{dy'}{dx} = \frac{1}{c} \sqrt{1 + y'^2}$$

"بفصل المتغيرات والتكامل"

$$\int \frac{dy'}{\sqrt{1 + y'^2}} = \frac{1}{c} \int dx$$

$$\sinh^{-1}(y') = \frac{1}{c} x + C_1 \quad \#$$

* $C_1 \rightarrow$ ثابت التكامل

$$\therefore \sinh^{-1}(y) = \frac{1}{c} x$$

$$\therefore y' = \sinh \frac{x}{c}$$

$$\int dy = \int \sinh \left(\frac{x}{c} \right) dx$$

$$y = c \cosh \left(\frac{x}{c} \right) + C_2 \quad \text{ثابت التكامل}$$

#

٣) المعادلة البارامترية

$$\cos \mu = \frac{dx}{ds}, \quad \cos \mu = \frac{c}{\sqrt{s^2 + c^2}}$$

$$\therefore \frac{dx}{ds} = \frac{c}{\sqrt{s^2 + c^2}}$$

$$\sin \mu = \frac{dy}{ds}, \quad \sin \mu = \frac{s}{\sqrt{s^2 + c^2}}$$

$$\therefore \frac{dy}{ds} = \frac{s}{\sqrt{s^2 + c^2}}$$

بالتكامل

$$y' = \sqrt{c^2 + s^2}$$

$$* \frac{dx}{d\mu} = \frac{dx}{ds} \cdot \frac{ds}{d\mu} = \cos \mu \cdot c \sec^2 \mu$$

$$\therefore \frac{dx}{d\mu} = c \sec \mu$$

$$x = c \ln \left(\frac{y}{c} + \frac{s}{c} \right)$$

$$x = c \ln \left(\frac{y+s}{c} \right)$$

#

$$x = c \ln (\sec \mu + \tan \mu) \quad \#$$

$$* \frac{dy}{d\mu} = \frac{dy}{ds} \cdot \frac{ds}{d\mu} = \sin \mu \cdot c \sec^2 \mu$$

$$\therefore \frac{dy}{d\mu} = c \sec \mu \tan \mu$$

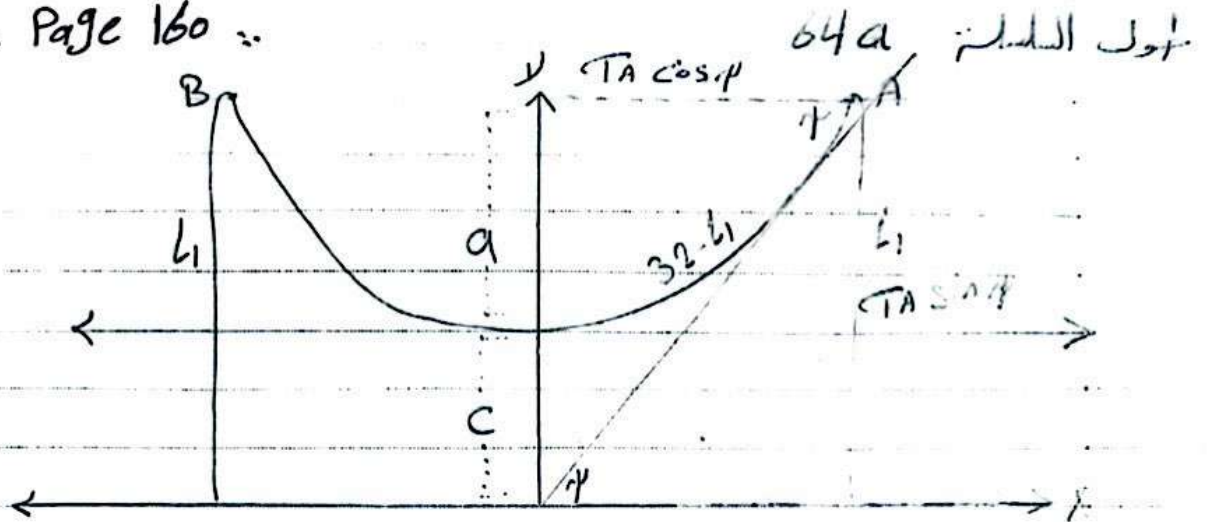
∴ Int.

$$y = c \sec \mu$$

#

/ /

أول السلسلة



$$J = L_1$$

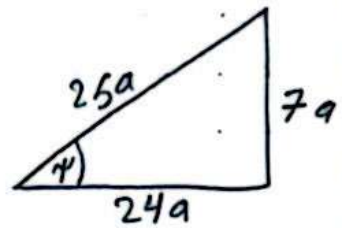
$$\Rightarrow V^2 = S^2 + C^2$$
$$(a+c)^2 = (31a-c)^2 + c^2$$
$$a^2 + 2ac + c^2 = 961a^2 - 62ac + c^2$$
$$960a^2 - 64ac + c^2 = 0$$
$$(40a - c)(24a - c) = 0$$
$$c = 40a, c = 24a$$

مرفوضه ۹۹ -

79 ✓

$$\Rightarrow L_1 = a + c = y$$

$$= a + 24a = \underline{25a}$$



$$\Rightarrow AB = 2X$$

$$= 2c \ln\left(\frac{s+y}{c}\right)$$

$$= 2 \cdot 24a \ln\left(\frac{7a+25a}{24a}\right)$$

$$= 48a \ln\left(\frac{4}{3}\right) \quad \#$$

* اصب الضغط على اى من الطرفين

$$\Rightarrow T_A = w y = \underline{25w}$$

$$\Rightarrow P^2 = (T_A + T_A \sin \phi)^2 + (T_A \cos \phi)^2$$

$$= T_A^2 + 2T_A^2 \sin^2 \phi + T_A^2 \sin^2 \phi + T_A^2 \cos^2 \phi$$

$$= T_A^2 (1 + 2\sin^2 \phi + \sin^2 \phi + \cos^2 \phi)$$

$$= 2T_A^2 (1 + \sin^2 \phi) = 2 \cdot (25w)^2 \cdot \left(1 + \frac{7}{25}\right)$$

$$= 1600$$

$$\therefore P = \sqrt{1600} = \underline{40}$$

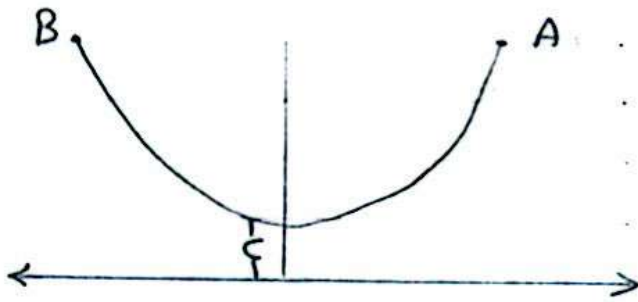
ALADIB

الخطوات

$$T_A = n T_0$$

$$\omega_y = n \omega_c$$

$$\therefore y = n c \# \quad s = L$$



$$AB = \frac{2L}{\sqrt{n^2-1}} \ln(\sqrt{n^2-1} + n) \quad \text{أثبت أنه} \dots$$

$$\Rightarrow y^2 = s^2 + c^2$$

$$n^2 c^2 = L^2 + c^2$$

$$L^2 = n^2 c^2 - c^2$$

$$L^2 = c^2 (n^2 - 1)$$

$$\therefore c^2 = \frac{L^2}{n^2 - 1}$$

$$\therefore c = \frac{L}{\sqrt{n^2 - 1}} \#$$

$$\therefore y = n c = \frac{nL}{\sqrt{n^2 - 1}} \#$$

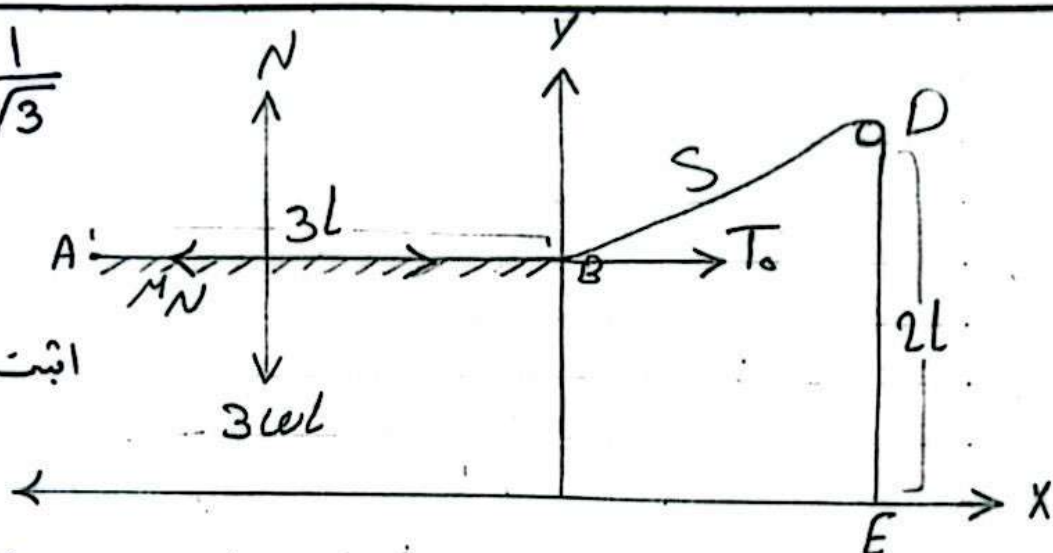
$$\Rightarrow AB = 2x = 2c \ln\left(\frac{s+y}{c}\right)$$

$$= \frac{2L}{\sqrt{n^2-1}} \ln\left[\left(L + \frac{nL}{\sqrt{n^2-1}}\right) \cdot \left(\frac{\sqrt{n^2-1}}{L}\right)\right]$$

$$= \frac{2L}{\sqrt{n^2-1}} \ln(\sqrt{n^2-1} + n)$$

#

معامل الاحتكاك $M = \frac{1}{\sqrt{3}}$
 $y = 2L$



أثبت أنه: $P = 2\sqrt{3}\omega L$

$$T_D = \omega y = 2\omega L$$

$$\Rightarrow N = 3\omega L$$

$$\Rightarrow T_0 = M_N = \omega C$$

$$= \frac{1}{\sqrt{3}} \cdot 3\omega L$$

$$= \frac{3}{\sqrt{3}} \omega L$$

$$\therefore C = \sqrt{3}L \#$$

$$\Rightarrow y^2 = S^2 + C^2$$

$$4L^2 = S^2 + 3L^2$$

$$\therefore S^2 = L^2 \Rightarrow \underline{S = L}$$

$$\tan \psi = \frac{S}{C} = \frac{L}{\sqrt{3}L} = \frac{1}{\sqrt{3}}$$

$$\therefore \psi = \underline{\underline{30^\circ}}$$

$$\Rightarrow P^2 = 2T_D^2 (1 + \sin \psi)$$

$$= 2T_D^2 (1 + \sin 30^\circ)$$

$$= \frac{3}{2} \cdot 2T_D^2 = 3T_D^2$$

$$\therefore P = \sqrt{3} T_D = \sqrt{3} \cdot 2\omega L$$

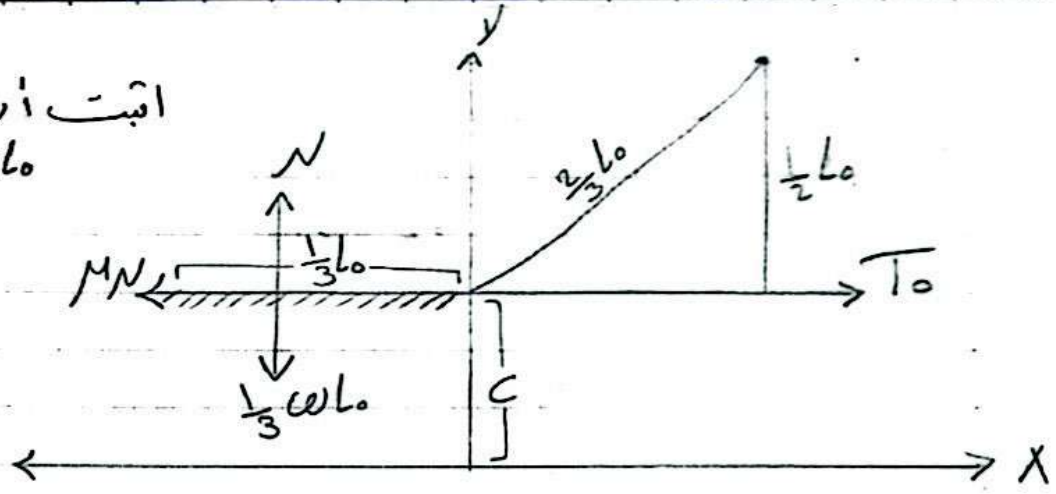
$$= 2\sqrt{3} \omega L \#$$

النتيجة

اكتب انه :

1) $C = \frac{1}{3} M L_0$

2) $M = \frac{7}{12}$



$S = \frac{2}{3} L_0$, $N = \frac{1}{3} \omega L_0$, $Y = \frac{1}{2} L_0 + C$

$\Rightarrow M_N = T_0 = \omega C$

$M(\frac{1}{3} \omega L_0) = \omega C$

$\therefore C = \frac{1}{3} M L_0$ (1) $\neq$

$Y = \frac{1}{2} L_0 + \frac{7}{36} L_0$

$Y = \frac{25}{36} L_0$

$\Rightarrow$ (2) , (1) no

$\Rightarrow Y^2 = S^2 + C^2$

$(\frac{1}{2} L_0 + C)^2 = (\frac{2}{3} L_0)^2 + C^2$

$\frac{1}{3} M L_0 = \frac{7}{36} L_0$

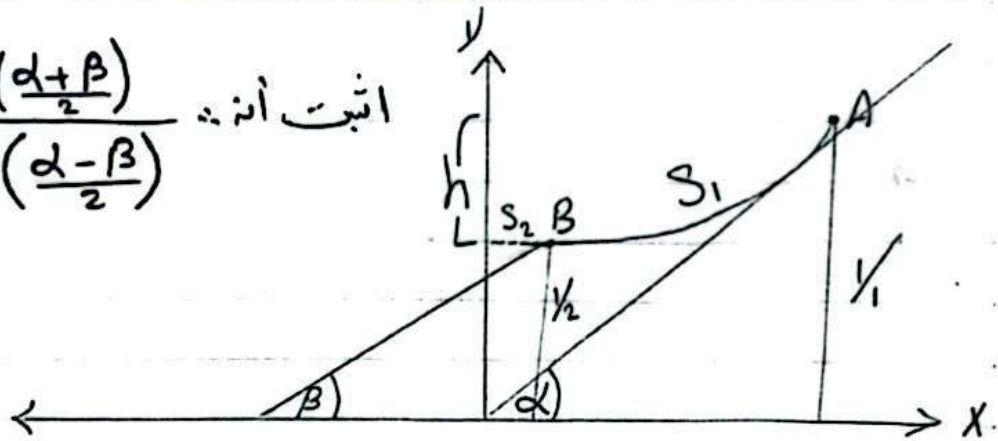
$\frac{1}{4} L_0^2 + C L_0 + C^2 = \frac{4}{9} L_0^2 + C^2$

$\therefore M = \frac{7}{36} \times 3 = \frac{7}{12}$

$C = \frac{4}{9} L_0 - \frac{1}{4} L_0 = \frac{16-9}{36} L_0$

$= \frac{7}{36} L_0$ (2)

$$h = \frac{L \sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \quad \text{اثبت أنه:}$$



$$h = y_2 - y_1, \quad L = S_1 - S_2, \quad S_1 = c \tan \alpha, \quad S_2 = c \tan \beta$$

$$\Rightarrow L = c (\tan \alpha - \tan \beta)$$

$$y_1 = c \sec \alpha, \quad y_2 = c \sec \beta$$

$$\Rightarrow h = c (\sec \alpha - \sec \beta)$$

$$\frac{L}{h} = \frac{c (\tan \alpha - \tan \beta)}{c (\sec \alpha - \sec \beta)} \Rightarrow h (\tan \alpha - \tan \beta) = L (\sec \alpha - \sec \beta)$$

$$h \left(\frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta} \right) = L \left(\frac{1}{\cos \alpha} - \frac{1}{\cos \beta} \right) \quad (\cos \alpha \cos \beta \text{ common})$$

$$h (\sin \alpha \cos \beta - \sin \beta \cos \alpha) = L (\cos \beta - \cos \alpha)$$

$$h (\sin (\alpha - \beta)) = L (2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right))$$

$$h (2 \sin \left(\frac{\alpha - \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)) = 2L \left(\sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right) \right)$$

$$h = L \frac{\sin (\alpha + \beta)}{\cos (\alpha - \beta)}$$

هي النقطة التي يتركز عندها الجسم

مركز: نقطة وصل إحداثي

ولذلك فإن، إحداثي مركز الثقل

1- لو في الفراغ هو $(\bar{x}, \bar{y}, \bar{z})$

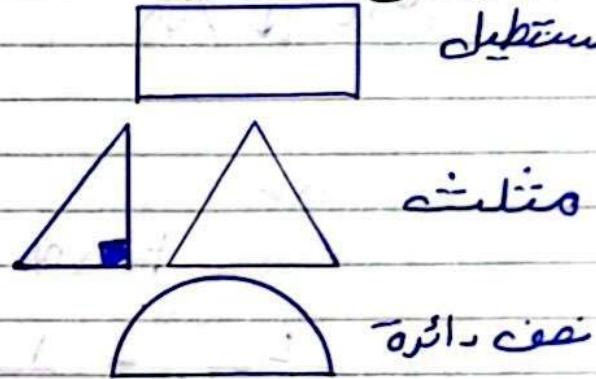
2- لو في مستوى xy هو $(\bar{x}, \bar{y})$

مركز الثقل

الأشكال الغير منتظمة

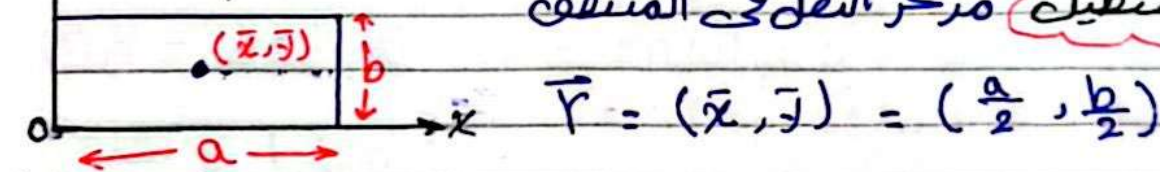


الأشكال المنتظمة

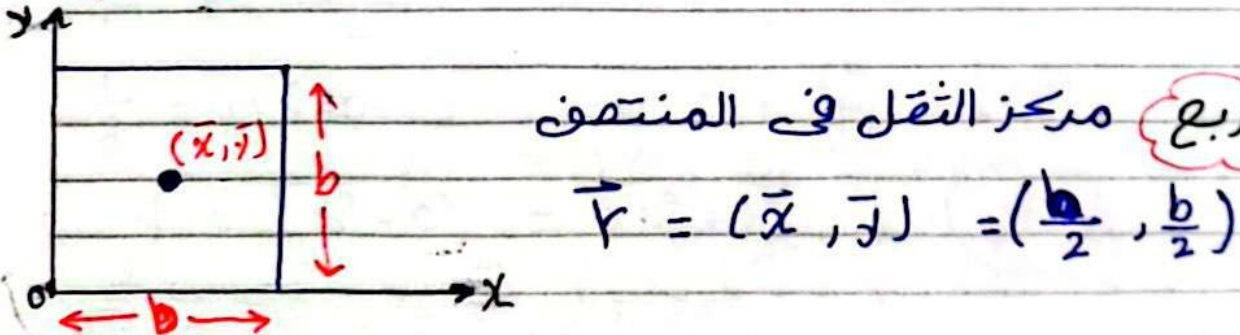


1) الأشكال المنتظمة:

المستطيل مركز الثقل في المنتصف

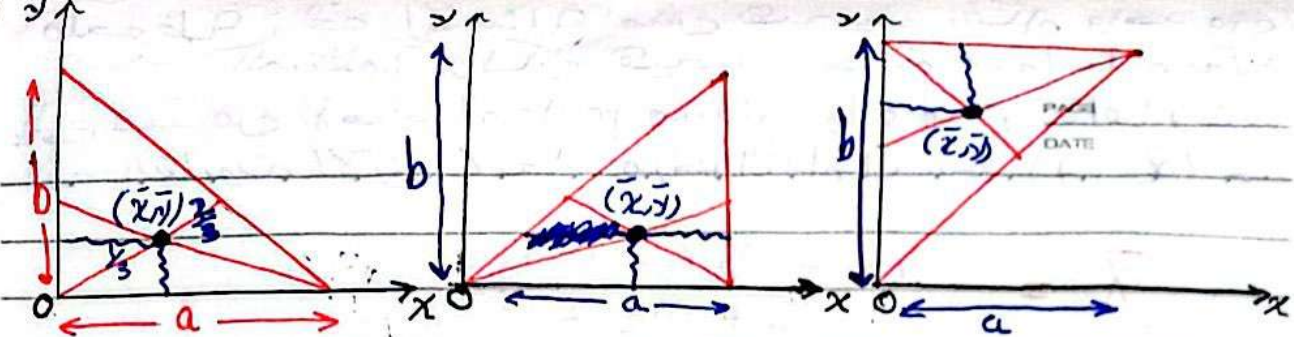


المربع مركز الثقل في المنتصف

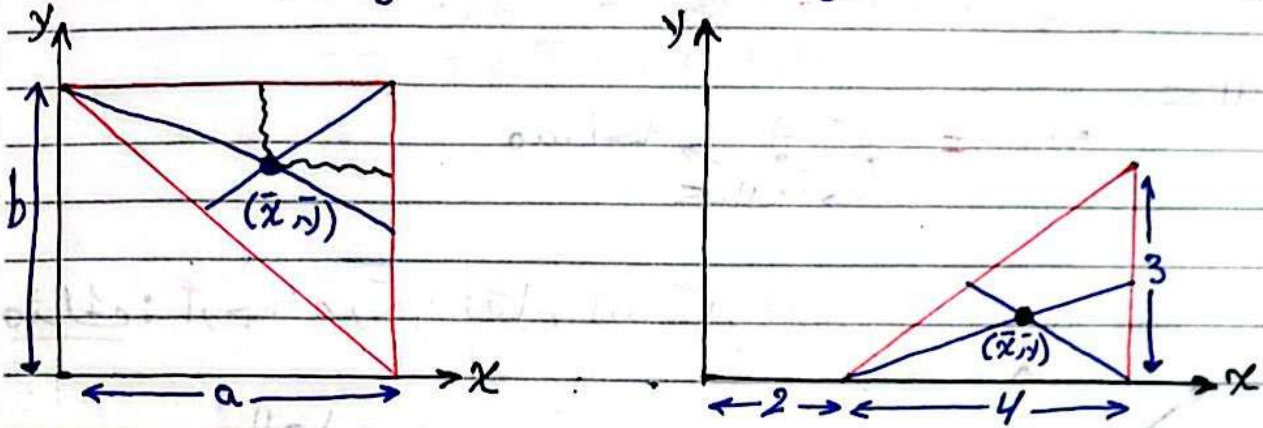


المثلث مركز الثقل عند نقطة التقاطع للمتوسطات
حيث أن نقطة تلاقي المتوسطات تقسم الطول

بنسبة $\frac{2}{3}$ ، $\frac{1}{3}$

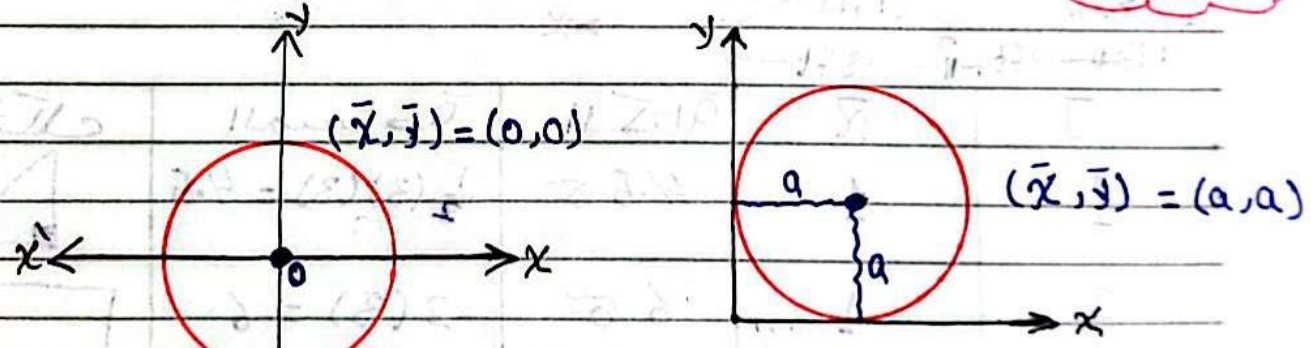


$$(\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{b}{3}\right) \quad (\bar{x}, \bar{y}) = \left(\frac{2a}{3}, \frac{b}{3}\right) \quad (\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{2b}{3}\right)$$

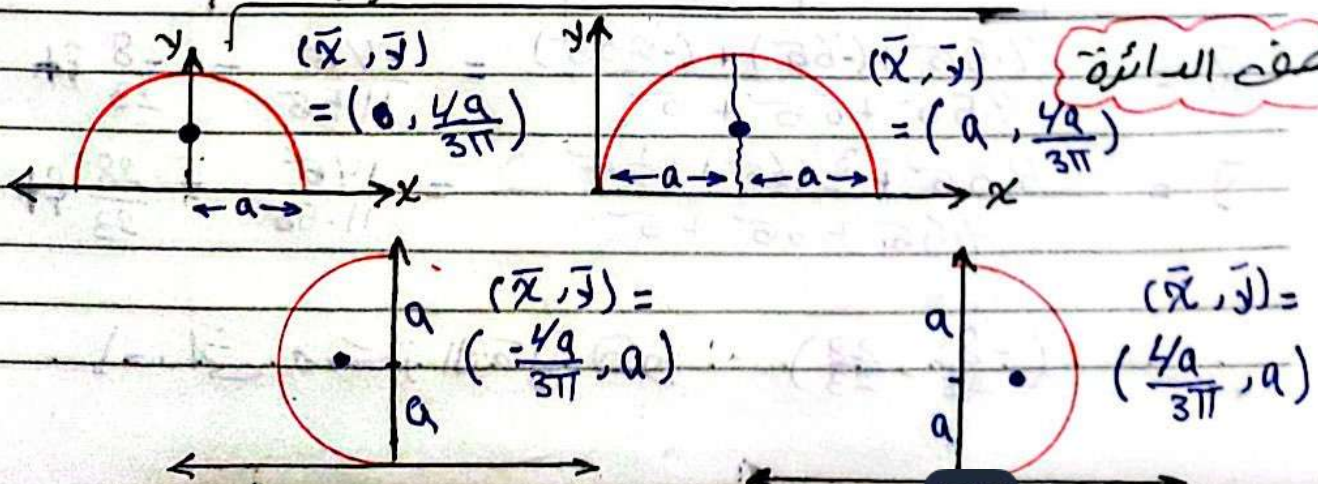


$$(\bar{x}, \bar{y}) = \left(\frac{2a}{3}, \frac{2b}{3}\right) \quad (\bar{x}, \bar{y}) = \left(2 + \frac{2 \times 4}{3}, \frac{3}{3}\right) = \left(\frac{14}{3}, 1\right)$$

الدائرة مركز ثقل الدائرة في المنتصف

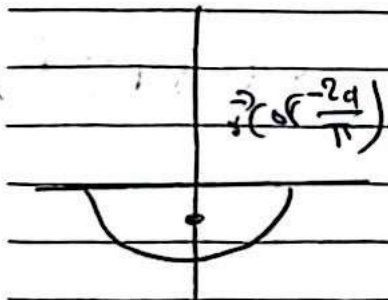
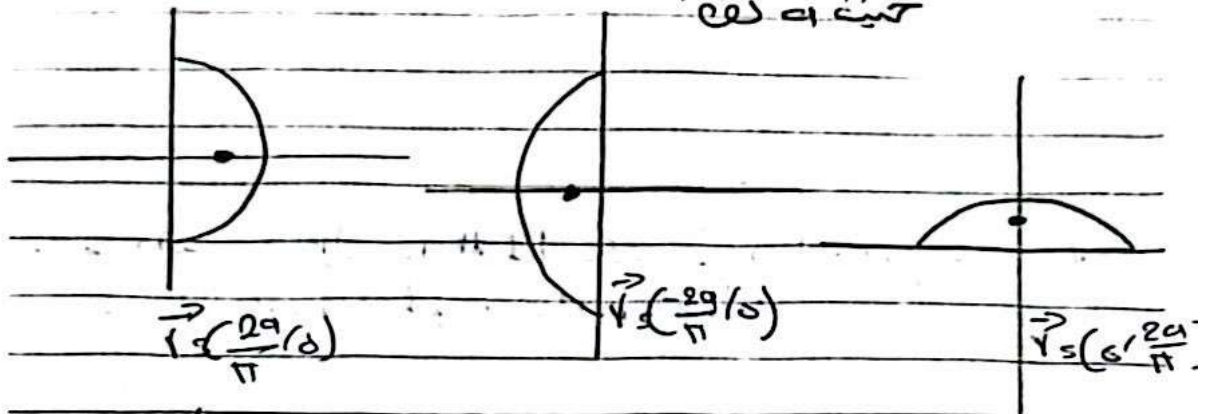


نصف الدائرة

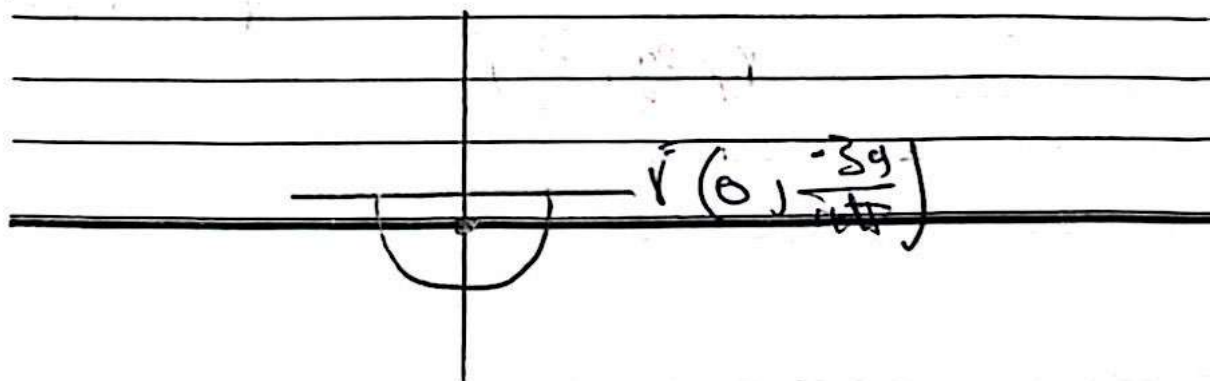
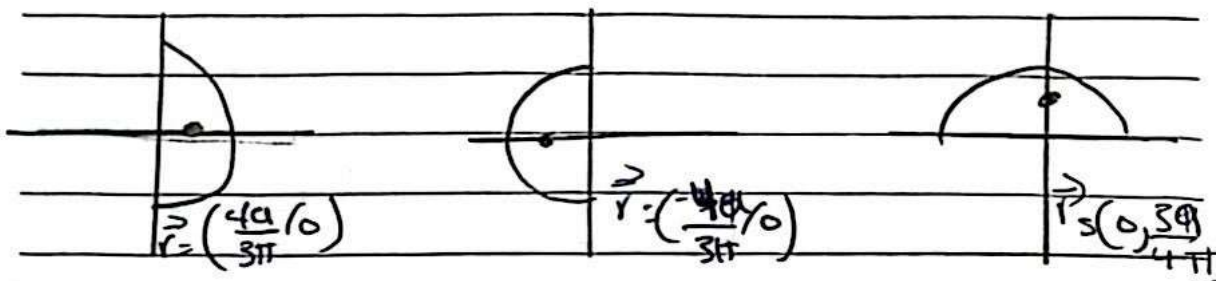


الدائرة - مركز ثقل الدائرة في منتصف

مركز ثقل نصف الدائرة على شكل سلك دائري (0, $\frac{2a}{\pi}$) حيث a نصف



مركز ثقل نصف الدائرة على شكل صفيحة (0, $\frac{4a}{3\pi}$) حيث a نصف



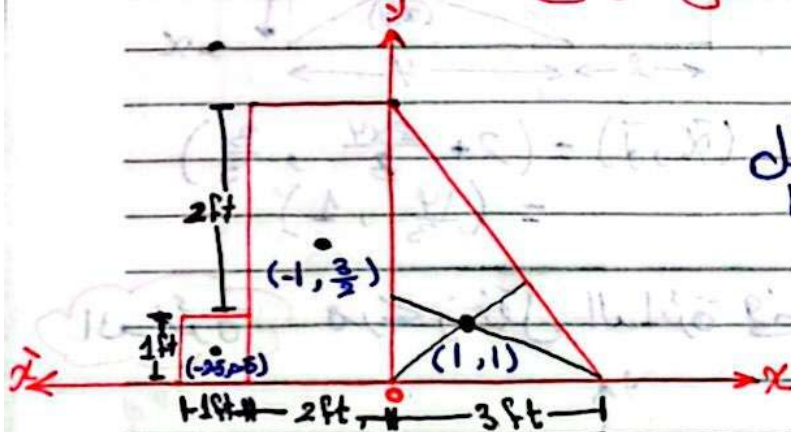
ملحوظة في الأسئلة مثل هيجيبه شكل واحد من الأشكال المنتظمة ولكن هيجيبه جسم متماسك يمكن تقسيمه إلى عدد من الأجزاء المعلوم مساحتها ومركز ثقلها وهنغوض فيه القانون الآتي لإيجاد مركز الثقل للجسم $(\bar{x}, \bar{y})$ حيث

$$\bar{x} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2 + m_3 \bar{x}_3 + \dots}{m_1 + m_2 + m_3 + \dots}$$

$$\bar{y} = \frac{m_1 \bar{y}_1 + m_2 \bar{y}_2 + m_3 \bar{y}_3 + \dots}{m_1 + m_2 + m_3 + \dots}$$

كتلة $m = \frac{A}{\rho}$ مساحة كثافة

مثال 1: أوجد مركز ثقل الشكل الآتي



الحل
سوف نقوم بتقسيم الشكل إلى 3 أشكال معلوم مساحتها ومركز ثقلها وهم مثلث و مستطيل ومربع .

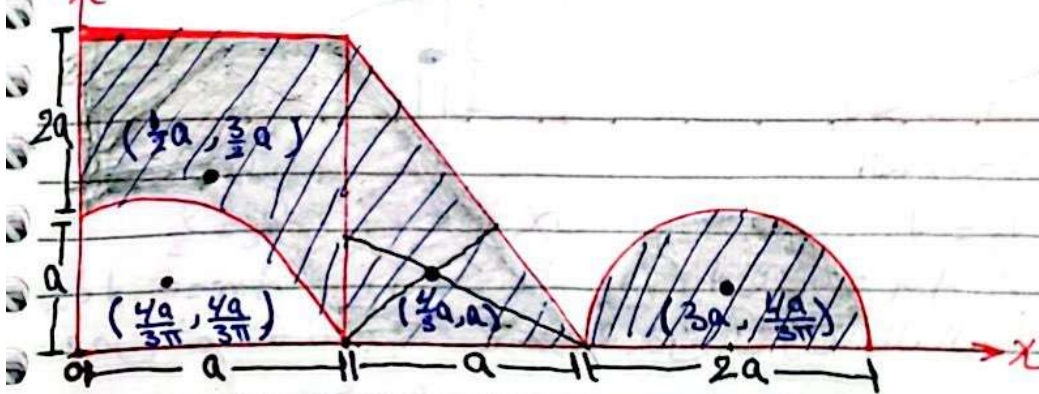
الشكل	المساحة	الكتلة	$\bar{x}$	$\bar{y}$
	$\frac{1}{2}(3)(3) = 4.5$	4.5 ρ	1	1
	$2(3) = 6$	6 ρ	-1	$\frac{3}{2}$
	$(1)^2 = 1$	ρ	-0.5	0.5

$$\bar{x} = \frac{4.5\rho + (-6\rho) + (-0.5\rho)}{4.5\rho + 6\rho + \rho} = \frac{-4\rho}{11.5\rho} = -\frac{8}{23} \text{ ft}$$

$$\bar{y} = \frac{4.5\rho + \frac{3}{2}(6\rho) + \frac{1}{2}\rho}{4.5\rho + 6\rho + \rho} = \frac{14\rho}{11.5\rho} = \frac{28}{23} \text{ ft}$$

إحداثي مركز الثقل هو $(-\frac{8}{23}, \frac{28}{23})$

مثال 2: أوجد مركز الثقل للشكل الآتي علماً بأن $\pi = 3.14$



الشكل	المساحة	الكتلة	$\bar{x}$	$\bar{y}$
	$\frac{1}{4} \pi a^2 = 0.785a^2$ $= -0.785a^2$	$-0.785a^2 \sigma$	$\frac{4a}{3\pi}$	$\frac{4a}{3\pi}$
	$a \cdot 3a = 3a^2$	$3a^2 \sigma$	$\frac{1}{2}a$	$\frac{3}{2}a$
	$\frac{1}{2} \cdot a \cdot 3a = \frac{3}{2}a^2$	$\frac{3}{2}a^2 \sigma$	$\frac{4}{3}a$	a
	$\frac{1}{2} \cdot \pi \cdot a^2 = 1.57a^2$	$1.57a^2 \sigma$	$3a$	$\frac{4a}{3\pi}$

$$\bar{x} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2 + m_3 \bar{x}_3 + m_4 \bar{x}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(-0.785a^2 \sigma) \frac{4a}{3\pi} + \frac{1}{2}a(3a^2 \sigma) + \frac{4}{3}a(\frac{3}{2}a^2 \sigma) + 3a(1.57a^2 \sigma)}{-0.785a^2 \sigma + 3a^2 \sigma + \frac{3}{2}a^2 \sigma + 1.57a^2 \sigma}$$

$$= \frac{7.88 a^3 \sigma}{5.3 a^2 \sigma} \approx \boxed{1.5 a}$$

$$\bar{y} = \frac{m_1 \bar{y}_1 + m_2 \bar{y}_2 + m_3 \bar{y}_3 + m_4 \bar{y}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(-0.785a^2 \sigma) \frac{4a}{3\pi} + \frac{3}{2}a(3a^2 \sigma) + a(\frac{3}{2}a^2 \sigma) + \frac{4a}{3\pi}(1.57a^2 \sigma)}{-0.785a^2 \sigma + 3a^2 \sigma + \frac{3}{2}a^2 \sigma + 1.57a^2 \sigma}$$

$$= \frac{6.34 a^3 \sigma}{5.3 a^2 \sigma} \approx \boxed{1.2 a}$$

٢) الأشكال غير المنتظمة: مثل المساحات المحددة بمنحنيات لتحديد مركز كتلة المفيدة نقوم بتقسيم المساحة إلى شرائح على هيئة مستطيلات رأسية أو أفقية وبأخذ إحدى الشرائح مساحة dA حيث:



الشريحة الأفقية

$$x_e = \frac{x}{2}, y_e = y$$

$$dA = x dy, dm = \sigma x dy$$

ومن ثم نحسب $\bar{x}$ من العلاقة التالية

$$\bar{x} = \frac{\int_0^a x_e dm}{\int_0^a dm}$$

$$\bar{y} = \frac{\int_0^b y_e dm}{\int_0^b dm}$$

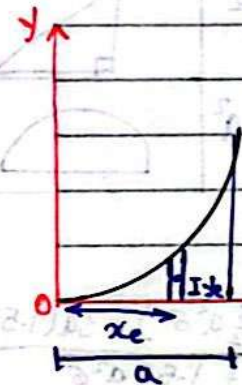
$$x_e = x, y_e = \frac{y}{2}$$

$$dA = y dx, dm = \sigma y dx$$

مثال أرسم المساحة المحددة بالمنحنى $y^2 = a^2 - x^2$ و المسقيمتين $x=a$ و $y=0$ ثم أوجد مركز ثقل تلك المساحة

$$y = \frac{x^2}{a}$$

Solution



$$x_e = x$$

$$y_e = \frac{y}{2}$$

$$dA = y dx, dm = \sigma y dx$$

$$\bar{x} = \frac{\int_0^a x_e dm}{\int_0^a dm}$$

$$\bar{x} = \frac{\int_0^a x y dx}{\int_0^a y dx} = \frac{\int_0^a x \cdot \frac{x^2}{a} dx}{\int_0^a \frac{x^2}{a} dx} = \frac{\int_0^a \frac{x^3}{a} dx}{\int_0^a \frac{x^2}{a} dx}$$

$$= \frac{\int_0^a x^3 dx}{\int_0^a x^2 dx} = \frac{\frac{1}{4} a^4}{\frac{1}{3} a^3} = \boxed{\frac{3}{4} a} \text{ cm}$$

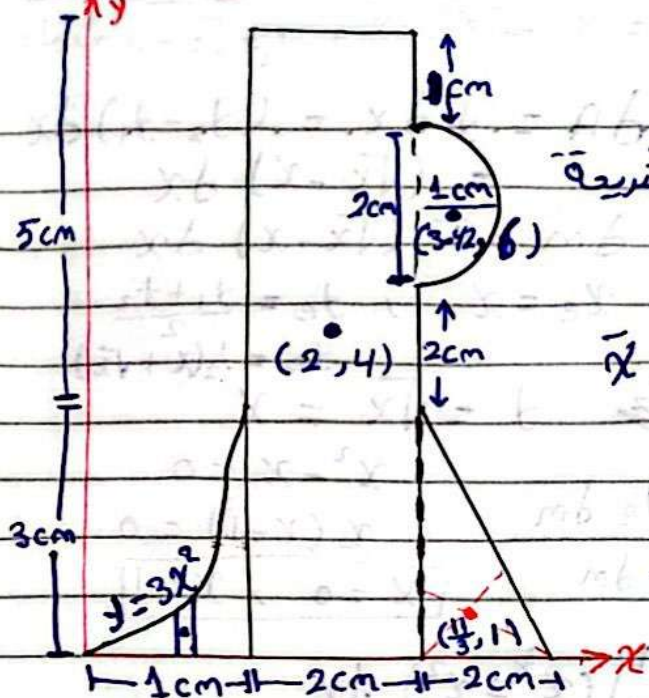
$$\bar{y} = \frac{\int_0^a y_e dm}{\int_0^a dm} = \frac{\int_0^a \frac{y}{2} \cdot y dx}{\int_0^a y dx} = \frac{1}{2} \frac{\int_0^a \frac{y^2}{2} dx}{\int_0^a \frac{x^2}{a} dx}$$

$$= \frac{1}{2a} \frac{\int_0^a x^4 dx}{\int_0^a x^2 dx} = \frac{1}{2a} * \frac{\frac{1}{5} a^5}{\frac{1}{3} a^3} = \boxed{\frac{3}{10} a} \text{ cm}$$

إحداثي مركز الثقل هو $(\frac{3}{4}a, \frac{3}{10}a)$



الاستقرار في السعي
رغم انعدام الشغف



الحل

أولاً: بنسبة للقطع المكافئ $y = 3x^2$ هناخذ شريحة

$$x_e = x, y_e = \frac{1}{2}$$

$$\bar{x} = \frac{\int_0^1 x_e dm}{\int_0^1 dm} = \frac{\int_0^1 x \cdot \frac{1}{2} dx}{\int_0^1 \frac{1}{2} dx} = \frac{\int_0^1 \frac{1}{2} x dx}{\int_0^1 \frac{1}{2} dx} = \left[\frac{\frac{1}{4} x^2}{\frac{1}{2} x} \right]_0^1 = \frac{3}{4}$$

$$\bar{y} = \frac{\int_0^1 y_e dm}{\int_0^1 dm} = \frac{\int_0^1 \frac{1}{2} y^2 dx}{\int_0^1 \frac{1}{2} dx}$$

$$= \frac{\int_0^1 \frac{9}{2} x^4 dx}{\int_0^1 \frac{1}{2} dx} = \frac{3}{2} \left[\frac{\frac{1}{5} x^5}{\frac{1}{2} x^3} \right]_0^1$$

$$= \frac{3}{2} * \frac{3}{5} = \frac{9}{10} = 0.9 \text{ cm}$$

الشكل	المساحة	الكتلة	$\bar{x}$	$\bar{y}$
	1 cm^2	6	0.75	0.9
	$2 \times 8 = 16 \text{ cm}^2$	16	2	4
	$\frac{1}{2} \pi r^2 = \frac{3.14 \times 1}{2} = 1.57 \text{ cm}^2$	1.57	3.42	6
	$\frac{1}{2} \times 2 \times 3 = 3 \text{ cm}^2$	3	3.67	1

$$\bar{x} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2 + m_3 \bar{x}_3 + m_4 \bar{x}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{0.75 + 2(16) + 3.42(1.57) + 3.67(3)}{1 + 16 + 1.57 + 3} = 2.28 \text{ cm}$$

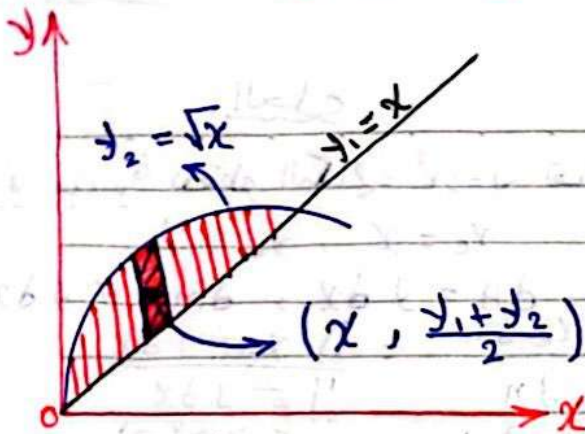
$$\bar{y} = \frac{m_1 \bar{y}_1 + m_2 \bar{y}_2 + m_3 \bar{y}_3 + m_4 \bar{y}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{0.9 + 4(16) + 6(1.57) + 3}{21.57} = 3.58 \text{ cm}$$

أحداث مركز الثقل (2.28, 3.58)

* مثال 3 احسبه مركز الثقل للشكل الآتي في المساحة بين

المنحنى $y_2 = \sqrt{x}$ والمستقيم $y_1 = x$



$$dA = y' dx = (y_2 - y_1) dx = (\sqrt{x} - x) dx$$

$$dm = \rho (\sqrt{x} - x) dx$$

$$x_c = x, \quad y_c = \frac{y_1 + y_2}{2} = \frac{1}{2}(x + \sqrt{x})$$

$$y = \sqrt{x} = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, \quad x = 1$$

$$\bar{x} = \frac{\int_0^1 x_c dm}{\int_0^1 dm}, \quad \bar{y} = \frac{\int_0^1 y_c dm}{\int_0^1 dm}$$

$$\bar{x} = \frac{\int_0^1 x (\sqrt{x} - x) dx}{\int_0^1 (\sqrt{x} - x) dx} = \frac{\int_0^1 (x^{3/2} - x^2) dx}{\int_0^1 (x^{1/2} - x) dx}$$

$$= \frac{[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3]_0^1}{[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2]_0^1} = \frac{2}{5}$$

$$\bar{y} = \frac{\int_0^1 (\frac{y_1 + y_2}{2}) (\sqrt{x} - x) dx}{\int_0^1 (\sqrt{x} - x) dx} = \frac{\int_0^1 (\frac{\sqrt{x} + x}{2}) (\sqrt{x} - x) dx}{\int_0^1 (\sqrt{x} - x) dx}$$

$$= \frac{1}{2} * \frac{\int_0^1 (x - x^2) dx}{\int_0^1 (\sqrt{x} - x) dx} = \frac{1}{2} * \frac{[\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1}{[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2]_0^1} = \frac{1}{2}$$

بإحداثي مركز الثقل $(\frac{2}{5}, \frac{1}{2})$

* مثال 4 أوجد مركز ثقل الشكل

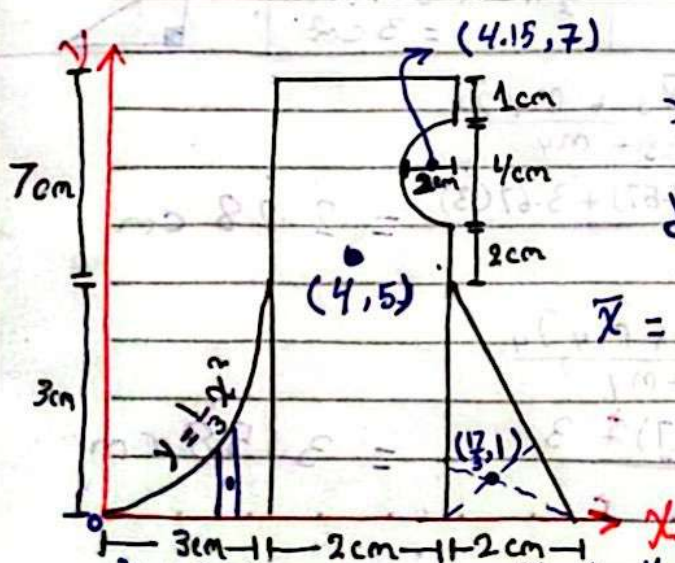
أولا بنسبة $y = \frac{1}{3}x^2$ للقطع المكافئ

$$x_c = x, \quad y_c = \frac{y}{2}$$

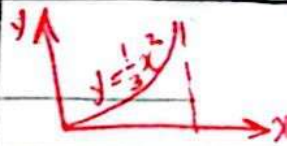
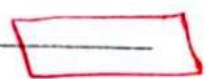
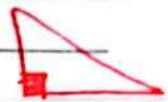
$$dA = y dx, \quad dm = \rho y dx$$

$$\bar{x} = \frac{\int_0^3 x y dx}{\int_0^3 y dx} = \frac{\int_0^3 \frac{1}{3} x^3 dx}{\int_0^3 \frac{1}{3} x^2 dx}$$

$$= \frac{[\frac{1}{12} x^4]_0^3}{[\frac{1}{9} x^3]_0^3} = \frac{9}{4} = 2.25 \text{ cm}$$



$$\bar{y} = \frac{\int_0^3 \frac{y^2}{2} dx}{\int_0^3 y dx} = \frac{1}{2} * \frac{\int_0^3 \frac{1}{3} x^4 dx}{\int_0^3 \frac{1}{3} x^2 dx} = \frac{1}{6} * \frac{[\frac{1}{15} x^5]_0^3}{[\frac{1}{9} x^3]_0^3} = \frac{9}{10} = 0.9 \text{ cm}$$

$\bar{y}$	$\bar{x}$	الكتلة	المساحة	الشكل
0.9	2.25	36	3 cm ²	
5	4	20.6	2 x 10 = 20 cm ²	
3+2+2 = 7	5 - $\frac{4+2}{3\pi}$ = 4.15	-6.286	$-\frac{1}{2}\pi r^2 = -\frac{1}{2} \times 3.14 \times 4$ = -6.28	
$\frac{3}{3} = 1$	5 + $\frac{2}{3} = \frac{17}{3}$	36	$\frac{1}{2} \times 2 \times 3 = 3$ cm ²	

$$dm = 6 \, y \, dx$$

$$m = \int_0^3 6 \cdot \frac{1}{3} x^2 \, dx = \frac{1}{3} 6 \left[\frac{1}{3} x^3 \right]_0^3 = 36$$

* كتلة القطع المائل $y = \frac{1}{3} x^2$

$$\bar{x} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2 + m_3 \bar{x}_3 + m_4 \bar{x}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{3(2.25) + 20(4) - 6.28(4.15) + 3(\frac{17}{3})}{3 + 20 - 6.28 + 3} = 3.94 \text{ cm}$$

$$\bar{y} = \frac{m_1 \bar{y}_1 + m_2 \bar{y}_2 + m_3 \bar{y}_3 + m_4 \bar{y}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{3(0.9) + 20(5) - 6.28(7) + 3(1)}{3 + 20 - 6.28 + 3} = 3.13 \text{ cm}$$

(3.94, 3.13) إحداثي مركز الثقل